Econ. 511b Spring 1999 George Hall and Chris Sims

"Traditional" vs. "Correct" Transversality Conditions, plus answers to problem set due 1/28/99*

1. Where the conventional TVC's come from

In a fairly wide class of growth models, the transversality conditions developed in previous lectures take on a simplified form, and under some side conditions can be converted into the transversality conditions that are usually taken to be standard. Consider an optimization problem of the following form:

$$\max_{\boldsymbol{C}_0^\infty, \boldsymbol{S}_0^\infty} E\left[\sum_{t=0}^{\infty} \beta^t U(C_t)\right] \tag{A1}$$

subject to

$$C_t \le f(K_t, K_{t-1}, L_t, \varepsilon(t)),\ t = 0, \dots, \infty\,. \tag{A2}$$

The Euler equations are

$$\partial C: \qquad D_C U_t = \lambda_t \tag{A3}$$

$$\partial L: \qquad D_L U_t = -\lambda_t D_3 f_t \tag{A4}$$

$$\partial K: \qquad \lambda_t D_1 f_t = -\beta E_t[\lambda_{t+1} D_2 f_{t+1}]\,. \tag{A5}$$

The transversality condition is

$$\limsup_{T\to\infty} \beta^T E[(D_C U_T - \lambda_T) dC_t + (D_L U_T + \lambda_T D_3 f_T) dL_T + \lambda_T D_1 f_T dK_T] \le 0\,. \tag{A6}$$

The Euler equations guarantee that the terms in dC_T and dL_T drop out, leaving

$$\limsup_{T\to\infty} \beta^T E[\lambda_T D_1 f_T dK_T] \le 0\,. \tag{A7}$$

A model that has an interpretation as a growth model will have $D_1 f_T < 0$, so that increasing K_t at t requires decreasing C_t. If the model satisfies $\hat{K}_T \ge 0$ for every feasible choice of K's, then $dK_T \ge -\bar{K}_T$ and in turn (A7) is less than

$$\limsup_{T\to\infty} \beta^T E[\lambda_T D_1 f_T \cdot (-\bar{K}_T)]\,. \tag{A8}$$

Thus a sufficient condition for transversality to hold is

$$\lim_{T\to\infty} \beta^T E[-\lambda_T D_1 f_T \bar{K}_T] = 0\,. \tag{A9}$$

Concavity and Convexity: A set S is **convex** if and only if whenever two points x_1 and x_2 are in S, $\alpha x_1 + (1-\alpha)x_2$ is also in S for any $\alpha \in [0,1]$. A function g is **concave** if and only if for any x_1 and x_2 arguments of g and any $\alpha \in [0,1]$, $g(\alpha x_1 + (1-\alpha)x_2) \geq \alpha g(x_1) + (1-\alpha)g(x_2)$. A function g is **quasi-concave** if for any c the set $\{x \,|\, g(x) \geq c\}$ is convex. All concave functions are quasi-concave. A **function g is convex** if and only if $-g$ is concave, and it is **quasi-convex** if and only if $-g$ is quasi-concave. These notes have usually assumed that constraints are in the form $g(x) \leq 0$, with g convex, but actually use only the fact that $\{x \,|\, g(x) \leq 0\}$ is convex. So the weaker assumption of quasi-convexity of g would have been enough.

In words, this says that the capital stocks (K might be a vector), valued at their marginal utility (and recognizing that this depends both on marginal utility of consumption and on the shadow price of capital $-D_1 f_T$) must increase slower than β^{-T}.

A very common special case is $D_1 f_T \equiv -1$. This corresponds to perfect substitutability of capital and consumption goods. It holds whenever the technology constraint is written with $C_t + K_t$ on the left, or when it has $C_t + I_t$ on the left and I_t is defined by $I_t = K_t - K_{t-1}(1-\delta)$. In this case the transversality condition becomes even simpler, reducing to

$$\lim_{T\to\infty} \beta^T E[\lambda_T K_T] = 0\,. \tag{A10}$$

This is the form of transversality condition you will most commonly see in the literature on growth and real business cycles.

Summarizing, a model in the form (A1)-(A2) in which $K \geq 0$ along on any feasible path and $K > 0$ along the optimal path, admits the simplified transversality condition (A9). If in addition the model makes capital and investment goods perfect substitutes ($D_1 f \equiv -1$), the further simplified TVC (A10) applies. Of course as usual the sufficient conditions for an optimum require the convexity, concavity and other conditions of the infinite-dimensional Kuhn-Tucker theorem in addition to the Euler equations and the TVC themselves.

2. The linear quadratic permanent income model revisited

Before proceeding to the problem set answer on this model, there is a technicality to be cleared up. The discussion of the model in the notes "First Order Conditions for Stochastic Problems: Examples" doesn't take note of the fact that the model as described does not fit the assumptions of the infinite-dimensional Kuhn-Tucker theorem in the notes "Random Lagrange Multipliers and Transversality". The theorem assumes that all constraints are inequalities, and includes in the sufficient conditions the requirement that all Lagrange multipliers are non-negative. The model writes the budget constraint as an equality.

In fact, the theorem does apply when some constraints are equalities and the Lagrange multipliers on those constraints are unrestricted in sign, so long as the equality constraints are linear. To make the extension, you replace the linear equality constraint $g_t = 0$ with the equivalent two inequality constraints $g_t \leq 0$ and $-g_t \leq 0$. These two constraints then have two non-negative Lagrange multipliers, only one of which is non-zero at any date, according to the direction in which the constraint binds. Because the constraint is linear, both g_t and $-g_t$ are convex functions. Making the problem fit the framework of the theorem in this way turns out to be equivalent to simply letting the Lagrange multiplier on the linear equality constraint be either positive or negative, as in the discussion of the model in the last set of notes.

2.1. **Problem 1.** It may have been ambiguous what was "the above analysis" that you were supposed to reproduce. We will go through everything done with the model, though a legitimate interpretation was that you were asked just to redo the analysis of the linearized model with time-varying r.

The FOC's are

$$\partial C: \qquad U'_t = \lambda_t \tag{A11}$$

$$\partial H: \qquad \lambda_t = \beta(1+r_t)E_t[\lambda_{t+1}]\,, \tag{A12}$$

which reduce to

$$U'_t = \beta(1+r_t)E_t[U'_{t+1}]\,. \tag{A13}$$

Notice that though (A11) and (A12) are different from the corresponding Euler equations (5) and (6) in the problem set notes, after elimination of λ, the equation (A13) that results is the same equation that emerges after elimination of λ from (5) and (6) in the notes.

First we specialize to the case of constant $1 + r = \beta^{-1}$, quadratic utility, i.i.d. Y. We can again derive from the FOC's the condition that C_t is a martingale, and solving forward to find a non-explosive solution for H leads to

$$H_t = \frac{\beta}{1-\beta}(C_t - \bar{Y}) \tag{A14}$$

or

$$C_t = rH_t + \bar{Y}\,. \tag{A15}$$

It can be checked that this implies both C_t and H_t are martingales.

The transversality condition is

$$\limsup_{T\to\infty} \beta^t E[(1 - \bar{C}_T - \lambda_T)dC_T - \lambda_T dH_T] \leq 0\,. \tag{A16}$$

Here the FOC (A11) guarantees that the term in dC_T above drops out. The same reasoning as in the notes tells us that in the solution $\bar{C}_t$ crosses above the satiation level infinitely often, forcing λ_T to be less than, say, $-\delta < 0$ infinitely often. It is then feasible to make dH_T positive at those dates on which $\lambda_T <= \delta$, and since doing so at any one date sets off an upward explosion at the rate $(1+r)^T$ in H_T, this allows us to make the lim sup in (A16) positive. This shows that the sufficient conditions for an optimum are not met at the non-explosive solution.

Now consider linearizing the more general model. Coupling the budget constraint (20) in the notes with (A13) gives us

$$\begin{bmatrix} \bar{U}''\beta(1+\bar{r}) & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} dC_{t+1} \\ dH_{t+1} \end{bmatrix} = \begin{bmatrix} \bar{U}'' & 0 \\ 0 & 1+\bar{r} \end{bmatrix} \begin{bmatrix} dC_t \\ dH_t \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \eta_{t+1} + \begin{bmatrix} \beta\bar{U}' & 0 \\ \bar{H} & 1 \end{bmatrix} \begin{bmatrix} dr_t \\ dY_{t+1} \end{bmatrix} . \tag{A17}$$

This leads to

$$A = \Gamma_0^{-1}\Gamma_1 = \begin{bmatrix} \beta^{-1}(1+\bar{r})^{-1} & 0 \\ -\beta^{-1}(1+\bar{r})^{-1} & 1+\bar{r} \end{bmatrix} . \tag{A18}$$

While this A is not exactly the same as that in (17) in the problem set notes, it is also triangular in structure, so the process of verifying existence and uniqueness is exactly parallel to that already discussed in the notes.

3. The simple growth model

3.1. **Problem 2.** The Euler equations for the problem as in the problem handout are

$$\partial C: \qquad D_C U_t = \lambda_t \tag{A19}$$

$$\partial L: \qquad D_L U_t = -\lambda_t D_3 f_t \tag{A20}$$

$$\partial K: \qquad \lambda_t = \beta E_t[\lambda_{t+1} D_K f_{t+1}] . \tag{A21}$$

Because the model falls in the simplest category of the previous section, with $D_1 f_t \equiv -1$, the TVC (we now know from the previous section) can be written in the form

$$\lim_{T\to\infty} \beta^T \lambda_T K_T = 0 . \tag{A22}$$

3.2. **Problem 3.** In this special case the Euler equations become

$$\partial C: \qquad \frac{1}{C_t} = \lambda_t \tag{A23}$$

$$\partial K: \qquad \lambda_t = \beta E_t[\lambda_{t+1}\alpha A_{t+1} K_t^{\alpha-1}] . \tag{A24}$$

We can then eliminate λ to obtain

$$\frac{1}{C_t} = \beta E_t\left[\frac{\alpha A_{t+1} K_t^{\alpha-1}}{C_{t+1}}\right] . \tag{A25}$$

The TVC becomes

$$\lim_{T\to\infty} \beta^T \frac{K_T}{C_T} = 0 . \tag{A26}$$

With K_t/C_t constant, the TVC is certainly satisfied.

Turning our attention to (A25), observe that from the technology constraint (22) in the problem statement notes,

$$A_{t+1}K_t^{\alpha-1} = \frac{K_{t+1} + C_{t+1}}{K_t}\,. \tag{A27}$$

Using this relationship in (A25), we arrive at

$$\frac{K_t}{C_t} = \beta\alpha E_t\left[\frac{K_{t+1}}{C_{t+1}} + 1\right]\,. \tag{A28}$$

But it is easy to see that this equation can be solved with K/C constant, so long as we set

$$\frac{K_t}{C_t} \equiv \frac{\alpha\beta}{1-\alpha\beta}\,. \tag{A29}$$

Thus this constant K/C ratio solves the Euler equations and and the satisfies the TVC. To finish the argument that it is the solution, observe that, because $\log C$ is twice differentiable and has second derivative $-1/C^2$ which is negative everywhere, it is concave. Also K^α is concave, for the same reason, and therefore $g(C_t, K_t, K_{t-1}, A_t) = C_t + K_t - A_t K^\alpha$ is convex, as required. This last step relies on two properties of concave and convex functions:

i. if g is concave, $-g$ is convex;
ii. linear combinations of concave functions are concave.

3.3. **George Hall's TVC.** George in class displayed as the TVC

$$\beta^T \frac{\alpha A K_T^{\alpha-1}}{K_{T+1} - AK_T^\alpha} \xrightarrow[T\to\infty]{} 0\,. \tag{A30}$$

There is no "E" operator here because A is non-random, so there is no uncertainty. The denominator of this expression is, from the technology constraint, just C_{T+1}. But then the Euler equation (A25), with the E's removed to reflect the absence of uncertainty, asserts exactly that the TVC's in (A30) and (A26) are the same.